Lecture 5. Linear transformations and their matrices

Def A function $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is called a linear transformation if it has the following properties:

(i) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for any $\vec{u}, \vec{v} \in \mathbb{R}^n$

(ii) $T(c\vec{v}) = cT(\vec{v})$ for any $c \in \mathbb{R}$, $\vec{v} \in \mathbb{R}^n$

Note In Math 313, we will use linear transformations to study various functions.

Def (1) The standard basis vectors of $\mathbb{R}^n$ are

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

* $\vec{e}_i$ has 1 as the i^{th} entry and 0 as all other entries

(2) The zero vector of $\mathbb{R}^n$ is the vector $\vec{0} \in \mathbb{R}^n$ whose entries are all zero.

Note Every vector in $\mathbb{R}^n$ can be written in terms of the standard basis vectors.

$$\vec{v} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \Longrightarrow \vec{v} = \begin{bmatrix} c_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ c_n \end{bmatrix}$$

$$\Longrightarrow \vec{v} = c_1\vec{e}_1 + c_2\vec{e}_2 + \cdots + c_n\vec{e}_n$$

Prop For every linear transformation T, we have $T(\vec{0}) = \vec{0}$.

pf $T(\vec{0}) = T(2\cdot\vec{0}) = 2T(\vec{0}) \Longrightarrow T(\vec{0}) = \vec{0}$

Def Given an $m\times n$ matrix A with columns $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n$ and a vector

$$\vec{v} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n,$$ their product is defined by

$$\boxed{A\vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n}$$

e.g. $A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 3 & 2 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$

$$\Rightarrow A\vec{v} = 1\cdot\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 3 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

Thm Given a linear transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, we have

$$\boxed{T(\vec{x}) = A\vec{x}}$$

where A is the $m\times n$ matrix with columns $T(\vec{e}_1), T(\vec{e}_2), \cdots, T(\vec{e}_n)$.

pf $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \Rightarrow \vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n$

$$\begin{aligned} T(\vec{x}) &= T(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n) \\ &= T(x_1\vec{e}_1) + T(x_2\vec{e}_2) + \cdots + T(x_n\vec{e}_n) \\ &= x_1T(\vec{e}_1) + x_2T(\vec{e}_2) + \cdots + x_nT(\vec{e}_n) = A\vec{x} \end{aligned}$$

Note (1) A is called the standard matrix of T.

(2) All coordinates of $T(\vec{x})$ are of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \quad \text{(with no constant terms)}$$

e.g. $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \Rightarrow A\vec{x} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 \\ 3x_2 \end{bmatrix}$

Ex Determine whether each function is a linear transformation.

(1) $T_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with $T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y^2 \\ x^3 \end{bmatrix}$

Sol The coordinates have power terms x^3 and y^2.

$\Rightarrow$ T_1 is $\boxed{\text{not a linear transformation}}$

(2) $T_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ with $T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+y \\ x-3y \\ 2y \end{bmatrix}$

Sol All coordinates are of the form $ax+by$

$\Rightarrow$ T_2 is $\boxed{\text{a linear transformation}}$

Note In fact, we have

$$T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

(3) $T_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with $T_3\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y+1 \\ 2y \end{bmatrix}$

Sol The first coordinate has a nonzero constant term.

$\Rightarrow$ T_3 is $\boxed{\text{not a linear transformation}}$

Note In fact, we have $T_3(\vec{0}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \vec{0}$

Ex Consider the linear transformation $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ with

$$T(\vec{e}_1) = \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad T(\vec{e}_2) = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \quad T(\vec{e}_3) = \begin{bmatrix} 0 \\ 3 \\ -2 \\ 0 \end{bmatrix}.$$

Find a formula for $T(\vec{x})$ with $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Sol The standard matrix of T is

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 3 \\ 1 & 0 & -2 \\ -1 & 1 & 0 \end{bmatrix}$$

with $T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3)$ as columns.

$$\Longrightarrow T(\vec{x}) = A\vec{x} = x_1 \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 3 \\ -2 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} 2x_1 + x_2 \\ 2x_2 + 3x_3 \\ x_1 - 2x_3 \\ -x_1 + x_2 \end{bmatrix}}$$

Note The entries of A yield the coefficients for x_1, x_2, x_3 in the formula.